The dust disk dynamics in week-nonlinear regime

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We investigate the problem of self-gravitating dust disk dynamics in a static state taking into account nonlinear effects. For this purpose Schrödinger-type equation including the mass conservation law is used for the whole description of hydrodynamic flows of self-gravitating dust. We have shown a purely hydrodynamic mechanism of ring formation in the radial direction by taking into account nonlinearity in the lowest order of expansion parameter, which determines an order of magnitude of flow.

1 Introduction

Self-gravitating systems are of great interest for investigation in astrophysics because of their widespread appearance [1, 2, 3]. Such systems are difficult for analytical analysis when we consider a lot of factors, and therefore the standard way of analysis is in the terms of density perturbations. As a result the problem may be entirely linearized, and it facilitates analysis, but in this way one cannot entirely investigate some specific effects caused by nonlinear dynamics of such systems. In this paper the problem is also considered in terms of perturbation theory, but it is possible to take into account nonlinear effects of dynamics in the lowest order by means of elimination of secular in time expansion terms. Schrödinger-type equation[1] representing combined description of Euler fluid dynamics together with the continuity equation is used

[1]Of course we consider this equation only as auxiliary one without Plank's constant and any "quantum sense".

for determination of potential hydrodynamic flow. Dynamics of rather outlying disk regions is discussed. These regions are situated far from the central massive object and the disk surface. Internal boundaries of these regions are predetermined by congruence condition of the self-gravitating dust potential and the compact body potential. The main aim of this paper is to consider a ring structure formation due to nonlinear hydrodynamic flow of self-gravitating dust. The internal region dynamics nearby the central object and the disk surface must be considered as the internal solution problem similarly to boundary layers problem.

2 The usage of Schrödinger-type equation in Euler fluid dynamics

Let us consider Schrödinger-type equation:

$$i\Psi_t + \alpha(t)\Delta\Psi - U(\mathbf{x},t)\Psi = 0. \tag{1}$$

Here Ψ is a dimensionless complex wave function, i is an imaginary unit, $\alpha = \alpha(t)$ is a certain dimensionless real function of time t, $U(\mathbf{x},t)$ is a real potential-like energy function, Δ is three-dimensional Laplacian.

Besides the (1) it is necessary to consider the complex conjugate equation:

$$-i\Psi_t^* + \alpha(t)\Delta\Psi^* - U(\mathbf{x},t)\Psi^* = 0.$$

Multiplying the previous equation by Ψ and (1) by Ψ^*, and subtracting the second expression from the first one, we get

$$\frac{\partial}{\partial t}|\Psi|^2 + \mathrm{div}\left(i\alpha|\Psi|^2\nabla\ln\Theta\right) = 0, \tag{2}$$

where $\Theta = \Psi^*/\Psi$. Dividing (1) and the conjugate equation by Ψ and Ψ^* correspondingly, and summing these expressions gives

$$i\frac{\partial}{\partial t}\ln\Theta + \alpha\frac{\Delta\Psi}{\Psi} + \alpha\frac{\Delta\Psi^*}{\Psi^*} - 2U = 0. \tag{3}$$

(2) can be interpreted as differential conservation law for density $|\Psi|^2$ of fluid moving with the velocity $\mathbf{v} = i\alpha\nabla\ln\Theta$.

For an arbitrary Euler flow $\mathbf{v}$ we have the following identity:

$$\mathbf{v}_t + (\mathbf{v}, \nabla)\mathbf{v} \equiv \frac{1}{2}\nabla|\mathbf{v}|^2 + [\mathbf{v} \times \mathrm{rot}\mathbf{v}] + \mathbf{v}_t. \tag{4}$$

Note that in our case the field $\mathbf{v}$ is potential: $\mathrm{rot}\mathbf{v} = 0$. For analysis of right-hand side of (4) one can use (3). As a result it follows the identity

$$\mathbf{v}_t + \frac{1}{2}\nabla|\mathbf{v}|^2 = \alpha^2\nabla\left[-2\frac{\Delta|\Psi|}{|\Psi|}\right] + 2\alpha\nabla U + \frac{\dot{\alpha}}{\alpha}\mathbf{v}.$$

This result through identity (4) can be combined into the following Euler's equation for the flow $\mathbf{v}$

$$\mathbf{v}_t + (\mathbf{v}, \nabla)\mathbf{v} = \alpha\nabla\left[-2\alpha\frac{\Delta|\Psi|}{|\Psi|} + 2U\right] + \kappa\mathbf{v}, \tag{5}$$

where $\kappa = -\dot{\alpha}/\alpha = \kappa(t)$ is a coefficient of a linear friction as it is called in fluid dynamics. Corresponding frictional force in the system $-\kappa\mathbf{v}$ can be regarded as a result of collisions between dust particles. In fact, this rather simplistic approach doesn't give an exact picture. Nevertheless, it allows to estimate the influence of a dissipation on a stationary dust distribution. Below we discuss the case $\kappa = \kappa_0 = \mathrm{const}$. Dependence κ on time can be interpreted as particles interaction to be changed in time, e. g. due to increasing particles in size .

Consider force per unit mass in the momentum equation in details. One can see from (5) the force is potential. In real fluid dynamics, however, right hand side in Euler equation in the presence of the field of a potential force is the following

$$\mathbf{F} = -\frac{1}{\rho}\nabla P - \nabla\phi, \tag{6}$$

where ρ is density of fluid or gas , P is pressure, ϕ is potential of force per unit mass is considered below to be Newtonian force. We investigate dust objects in

this paper. And the state equation for dust is well known as $p = 0$. Therefore, force per unit mass (6) consists of dust self-gravitation and it may include gravitation force from a massive object, which is nearby the dust. Then we find the relation between Euler fluid dynamics and Schrödinger-type equation

$$-2\alpha^2 \frac{\Delta|\Psi|}{|\Psi|} + 2\alpha U = -\phi. \tag{7}$$

3 The hydrodynamic equations for self-gravitating dust

The problem considered in this paper can be formulated in the following way. We investigate dust objects having the equation of state $P = 0$. And dust is in self-gravitation in terms of Euler fluid dynamics. So, we come to the system of equations for self-gravitating objects dynamics, which consists of Schrödinger-type equation (1), the concordance equation and the Poisson one, which is:

$$\Delta\phi = 4\pi G(\rho_0|\Psi|^2 + \sigma\delta(z) + M_0\delta(r)), \tag{8}$$

where ρ_0 - a characteristic density such that the function of density is

$$\rho = \rho_0|\Psi|^2.$$

The second z-direction δ-like source item in right hand side of the equation for potential describes a matter originally concentrated in thin disk with surface density $\sigma = \sigma(x, y, t)$, where x, y are Cartesian coordinates at the disk surface.

Now we shall make a nondimensionalization as following:

$$\tilde{\mathbf{r}} = \mathbf{r}/R_0, \quad \tau = t/T_0, \quad \Phi = \phi/\phi_0.$$

And make a notion : $\alpha(\tau) = \alpha_0 f(\tau)$, where

$$f(\tau) = \exp\{-\int_0^\tau \kappa(\tau')d\tau'\}$$

is a dimensionless function of time. The system of equations then reads

$$i\Psi_\tau + \varepsilon f(\tau)\tilde{\Delta}\Psi - W\Psi = 0, \tag{9}$$

$$-2\varepsilon^2 f^2(\tau)\frac{\tilde{\Delta}|\Psi|}{|\Psi|} + 2\varepsilon f(\tau)W = -\frac{\phi_0 T_0^2}{R_0^2}\Phi, \tag{10}$$

$$\tilde{\Delta}\Phi = \frac{4\pi G\rho_0 R_0^2}{\phi_0}|\Psi|^2, \tag{11}$$

where $\varepsilon = T_0\alpha_0/R_0^2$ - is the dimensionless characteristic parameter, estimating an order of magnitude of the dimensionless flow in the system:

$$\mathbf{V} = i\varepsilon f(\tau)\frac{\partial}{\partial\tilde{\mathbf{r}}}\ln\Theta,$$

$W = UT_0$ is the non-dimensional collective potential.

Below we omit ˜, implying the equations to be written in a dimensionless form.

In this paper we are interested in cases such that $\varepsilon << 1$ is a small parameter, i.e. the velocity of flow is small and the system in a gravitation field is about equilibrium .

Researching of (9)-(11) shows that for the equations to describe non-trivial situation we must use the conditions as follows:

$$\frac{\phi_0 T_0^2}{R_0^2} = \varepsilon, \quad \frac{4\pi G\rho_0 R_0^2}{\phi_0} = \mu = O(1),$$

where μ is the first-order constant to ε. From this it follows $\phi_0 = \varepsilon R_0^2/T_0^2, \quad \rho_0 = \varepsilon\mu/(4\pi G T_0^2)$, i.e. the dust density and the self gravitational potential are small and have the same order with respect to ε.

4 Approximate equations

Let us seek the solutions as power series in ε:

$$\Psi = \Psi_0 + \sum_{n=1}^{\infty}\varepsilon^n\Psi_n, \quad \Phi = \Phi_0 + \sum_{n=1}^{\infty}\varepsilon^n\Phi_n, \quad W = W_0 + \sum_{n=1}^{\infty}\varepsilon^n W_n.$$

Since small parameter ε in (9) and (10) is at the derivative of higher order, we can expect boundary layers in the system to appear. They could exist in the center of the field and at the surface of the disk. This boundary layers are connected with nonlinear mode not with viscosity. Outside of this boundary layers, i.e. far from the field center and the disk surface we use ordinary axial coordinate z. Nearby the disk surface we must use coordinate $Z = z/\varepsilon$. Note, δ-like source in (11) must be taken in account only in internal solution.

For external region we have the system of equations by substituting the expansions in equations in two first orders :

$$i\Psi_{0,\tau} = W_0\Psi_0; \qquad 2f(\tau)W_0 = -\Phi_0; \qquad \Delta\Phi_0 = \mu|\Psi_0|^2; \tag{12}$$

$$\begin{aligned} &i\Psi_{1,\tau} = W_0\Psi_1 + W_1\Psi_0 - \Delta\Psi_0, \\ &-2f^2(\tau)\frac{\Delta|\Psi_0|}{|\Psi_0|} + 2f(\tau)U_1 = \Phi_1, \qquad \Delta\Phi_1 = \mu(\Psi_0^*\Psi_1 + \Psi_1^*\Psi_0). \end{aligned} \tag{13}$$

Suppose the flow and the gravitational field in lowest order are stationary; then we have solution in this order:

$$\Psi_0 = C_0(\mathbf{r})\exp\left\{-i\int W_0(\mathbf{r},\tau)d\tau\right\}, \quad W_0 = -\frac{1}{2f(\tau)}\Phi_0,$$

where function Φ_0 is to be obtained from the Poisson equation:

$$\Delta\Phi_0 = \mu|C_0|^2. \tag{14}$$

After some simple manipulations, we arrive at the following solutions at the first order :

$$\Psi_1 = C_1(\mathbf{r})e^{-i\chi(r,\tau)} - ie^{-i\chi(\mathbf{r},\tau)}\int\limits_0^\tau \left[W_1(\mathbf{r},\tau')C_0(\mathbf{r}) - e^{i\chi(\mathbf{r},\tau')}f(\tau')\Delta\Psi_0\right]d\tau',$$

$$W_1 = -\frac{1}{2f(\tau)}\Phi_1 + f(\tau)\frac{\Delta|C_0|}{|C_0|}.$$

Here

$$\chi(r,\tau) = \int W_0(\mathbf{r},\tau)d\tau$$

Substituting these expressions in the equation for Φ_1 (13)we get:

$$\Delta\Phi_1 = \mu(C_0C_1^* + C_1C_0^*) + i\mu[C_0^*\Delta C_0 - C_0^*\Delta C_0]Q(\tau) - \mu H(\tau)\nabla\left(|C_0|^2\nabla\Phi_0\right) \tag{15}$$

Here

$$Q(\tau) = \int_0^\tau f(\tau')d\tau', \qquad H(\tau) = \int_0^\tau\int_0^{\tau'} \frac{d\tau''}{f(\tau'')}f(\tau')d\tau',$$

If $\kappa = \kappa_0 = \mathrm{const}$ then

$$Q(\tau) = \frac{1}{\kappa_0}(1 - e^{-\kappa_0\tau}), \quad H(\tau) = \tau\frac{1}{\kappa_0} - \frac{1}{\kappa_0^2}(1 - e^{-\kappa_0\tau}).$$

One can see that $Q(\tau)$ decays exponentially to a constant, whereas $H(\tau)$ increases linearly with time, i.e. corresponding component is secular, and for stable solution it should become zero. Hence, we come to the following: taking into account (14), functions $C_0(\mathbf{r})$ and $W_0(\mathbf{r})$ must obey the equations such that

$$\nabla\left(|C_0|^2\nabla\Phi_0\right) = 0, \tag{16}$$

$$\Delta\Phi_0 = \frac{\mu}{2}|C_0|^2. \tag{17}$$

Then equation (15) reduces to

$$\Delta\Phi_1 = \mu(C_0C_1^* + C_1C_0^*) + \frac{2\mu}{\kappa_0}\mathrm{div}[|C_0|^2\nabla\Theta_0]\left(1 - e^{-\kappa_0\tau}\right), \tag{18}$$

where $C_1(\mathbf{r})$ can be obtained from stationary condition at the next order of expansion , and $\Theta_0 = (i/2)\ln(C_0^*/C_0)$ - is still an arbitrary function. The solution for W_1 follows from the first equation in (13).

The interpretation of obtained equations follows from the expression for the flow velocity in the first order

$$\mathbf{V}_1 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{v}_1 = \varepsilon\nabla\Phi_0, \qquad \mathbf{v}_2 = 2\varepsilon f(\tau)\nabla\Theta_0(\mathbf{r}) = 2\varepsilon e^{-\kappa_0\tau}\nabla\Theta_0(\mathbf{r}).$$

Right hand member in (15) is determined by the source of mass, which is associated with the second flow in the system. This flow tends to a fixed space distribution as $\tau \to \infty$. Equation (16) is the law of conservation of mass for the flow $\mathbf{v}_1$. Suppose sources of mass do not exist; then the second addend in right hand side of (15) becomes zero

$$\mathrm{div}[|C_0|^2 \nabla \Theta_0] = 0. \tag{19}$$

If not, we must explicitly write down the source of mass by means of join of external and internal solution. The first flow is stationary and it is associated in first order with stationary fall in the field Φ_0. Existence of dissipation leads to fall of particles with the fixed velocity $\mathbf{v}_1$ instead of falling with the acceleration $\mathbf{g} = -\nabla \Phi_0$.

5 Axial-symmetric solutions

Our aim is to consider models with axial symmetry. Take cylindrical polar coordinates r, z, φ implying dependence functions of the system on r and z

$$\Phi_0(r, z) = u(r)h(z), \quad \mathcal{R}(r, z) = p(r)h(z).$$

Thus one can find the following equations for $u(r),\ p(r),\ h(z)$:

$$\frac{u''}{u} + \frac{1}{r}\frac{u'}{u} + \frac{h''}{h} = \frac{\mu}{2}\frac{p}{u}, \qquad \frac{u'}{u}\frac{p'}{p} + \frac{(h')^2}{h^2} + \frac{\mu}{2}\frac{p}{u} = 0.$$

Separation of variables implies

$$h(z) = h_0 e^{-\lambda z}.$$

We should suppose dust density and potential to decay while moving off the disk in the line of $z \to +\infty$ as well as $z \to -\infty$. Thus if $\lambda > 0$, we have $h(z) = h_0 e^{-\lambda |z|}$ far from the disk surface $z = 0$.

So we obtain equations for u and p

$$u'' + \frac{1}{r}u' + \lambda^2 u = \frac{\mu}{2}p, \tag{20}$$

$$\frac{u'}{u}\frac{p'}{p} + \lambda^2 + \frac{\mu}{2}\frac{p}{u} = 0. \tag{21}$$

Let us seek the solution for p as: $p(r) = q(r)u'(r)$. Substituting p in above form into equation (21) and using (20) yield

$$q' - \frac{1}{r}q + \mu q^2 = 0.$$

This equation is easy to solve and general solution is

$$q(r) = \frac{2}{\mu}\frac{r}{r^2 + Q_0},$$

here Q_0 - integral constant. Finally we get

$$p(r) = \frac{2}{\mu}\frac{r}{r^2 + Q_0}u'(r), \tag{22}$$

where $u(r)$ now follows the equation

$$u'' + \frac{Q_0}{r(r^2 + Q_0)}u' + \lambda^2 u = 0. \tag{23}$$

Function $p(r)$ follows, correspondingly, the equation

$$p'' + \frac{2r^2 - Q_0}{r(r^2 + Q_0)}p' + \lambda^2 p = 0. \tag{24}$$

At small Q or large r the equation for $u(r)$ is similar to oscillation equation with wave number λ and, hence, $u(r)$ changes the sign quasi-periodically as well as its derivative.

So we can establish behavior of dust distribution. The definition of $p(r)$ implies $p(r) > 0$. The density becomes zero together with the gradient of

potential according to (22). Thus it follows that disk is partitioned on rings which are separated by thin gaps. Analysis of equations for flow shows that dust from certain ring does not penetrate the bounders. The gradients of potential in adjacent rings, however, have unlike signs due to its quasi-periodic behavior. But if we require continuity of density and its derivative at boundary points, we immediately obtain that density must attain a negative value along with u'. Therefore, the derivative of density has discontinuity at boundaries of rings and in each ring with constant sign of u' we should choose the sign and the magnitude of Q to satisfy requirement $p(r) > 0$.

Generally boundary conditions for calculation of ring parameters come to continuity of potential and its derivative at ring boundaries (second derivative is discontinuous). It follows from equality of forces operating at disk boundaries.

Let $r_i, \quad i = 1, 2 \ldots$ be boundary points in which $u'|_{r_i} = 0$. Then equations for our model are:

$$u_j'' + \frac{Q_j}{r(r^2 + Q_j)} u_j' + \lambda_j^2 u_j = 0, \quad r \in [r_j, r_{j+1}]$$

with boundary conditions

$$\begin{aligned}
&u_j'(r_j) = 0, \quad u_j'(r_{j+1}) = 0, \\
&u_{j-1}(r_j) = u_j(r_j), \quad u_{j+1}(r_{j+1}) = u_j(r_{j+1}), \\
&\pi\mu \int\limits_{r_j}^{r_{j+1}} \frac{r^2}{(r^2 + Q_j)} u_j'(r) dr = \sigma_j.
\end{aligned}$$

Here σ_j is a constant surface density in j ring.

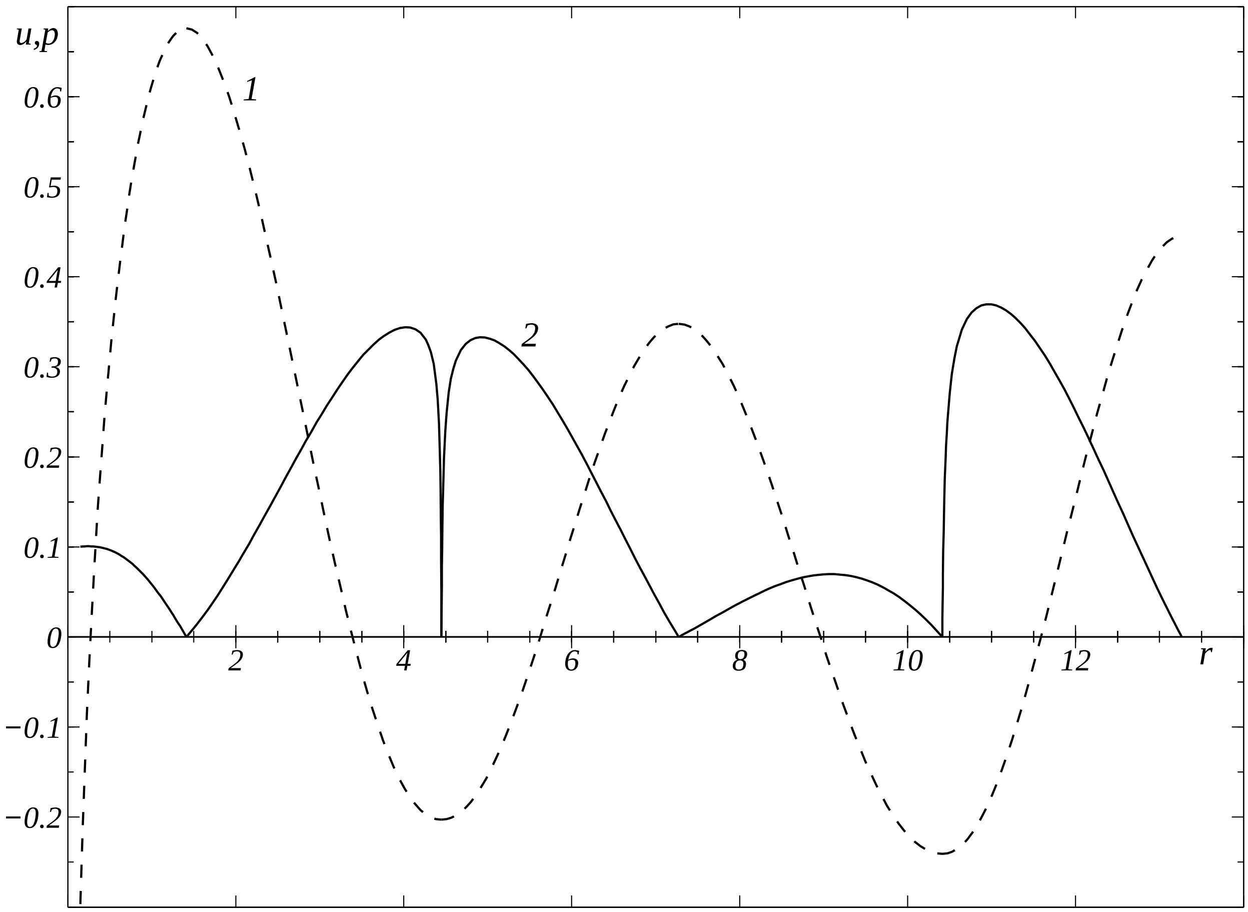

Fig.1. 1 - potential $u(r)$, 2 - density $p(r)$. Ring parameters : $\lambda = 1$, $\mu = 2$, $Q_0 = 10,\ Q_1 = -19.85,\ Q_2 = -19.7,\ Q_3 = -155,\ Q_4 = -108.$

Figure 1 illustrates the different possible solutions satisfying the boundary conditions.

6 Discussion

We have investigated the formation of rings with too narrow gaps between them as $t \to \infty$. And the width of gaps are considerably smaller then disk rings one. The masses of rings and distribution of density are determined by

the coefficient Q, that can be unique for each ring. Obtained solutions can be associated with dust distribution in real disk systems. However, to describe systems like internal Saturn's rings we must use other methods because our approach is not suitable for internal regions. Radial orbital flow induced by the central mass is the main component of dynamics for internal rings. Nearby the planet the radial velocity is great but we assume the mechanism of ring formation to be the same and it can be modified by corrections connected with the main orbital flow. And it will be the object of another paper.

References

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